Tutorial Paper

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How Unstable is an Unstable System?

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***Abstract*—Control theorists have extensively studied stability as well as relative stability of LTI systems. In this research paper we attempt to answer the question. How unstable is an unstable system? In that effort we define instability exponents dealing with the impulse response. Using these instability exponents, we characterize unstable systems.**

***Index Terms*—Stability, Relative Stability, Instability Exponent**

I. Introduction

Dynamical systems arise as models of natural and artificial phenomena evolving in time. Utilizing the principle of parsimony, linear time invariant (LTI) systems are utilized as models in various signal processing, control, and communication problems. Detailed theoretical results associated with such systems are derived and utilized in applications. One such theoretical concept deals with the stability of such systems.Bounded input, bounded output (BIBO) stability of LTI systems [1] is well studied. Traditionally, systems are classified as BIBO stable and BIBO unstable. Furthermore, relative stability of such systems explains how stable an LTI system is.

But, the binary classification (of stability) is very coarse and needs to be improved. Motivated by such observation, we investigate possible answers to the following question. How unstable is an unstable system? This paper is an effort to answer such a question. This paper is organized as follows. In section-II, we discuss some crude instability measures followed by section-III in which we consider discrete-time unstable LTI systems and associate an instability exponent with them. In section-IV, we consider continuous-time unstable systems and characterize them with the instability exponents. In section-V, we briefly consider state-space formulation of LTI systems. In section-VI, we outline the future research related to this research paper.

II. Crude Instability Measures

In the traditional literature on modern control systems stability and relative stability are well studied. In this paper, we propose to quantify the instability of discrete-time as well as continuous-time LTI systems. In such an effort, the following crude instability measures arise very naturally.

Category-A

- Number of poles in the right-half S-plane (continuous-time)
- Number of poles outside the unit circle (discrete-time)

Category-B

- Largest unstable pole in magnitude: right-half plane (continuous-time), outside unit circle (discrete-time).

Category-C

- Ratio of number of unstable poles to stable poles

All the above crude instability measures are based on the characteristic polynomial of unstable system [2]. For instance, the measure in Category-B can be determined by using an algorithm for computing the largest zero of a polynomial.The effective problem boils down to arriving at measures which summarize the information provided by the unstable poles, related to instability. Thus, this approach works on the frequency domain description of LTI system.

The above crude instability measures indirectly take into account, the impulse response of discrete-time as well as continuous-time LTI systems. In the following sections, we come up with measures of instability directly dealing with the impulse response of such system.

III. Discrete Time Unstable LTI Systems

In the following discussion, we first consider discrete LTI system. We effectively consider BIBO stability of an LTI system. It is well known that a discrete time LTI system is BIBO stable if and only if the impulse response is absolutely summable [3], i.e., if *h[k]* denotes the impulse response sequence, then the necessary and sufficient condition for BIBO stability is given by the condition

$$S = \sum_{k=-\infty}^{+\infty} \left| h[k] \right| < \infty \tag{1}$$

Now, in this paper, we are interested in understanding and characterizing how unstable is an unstable system, i.e., those systems for which

$$S = \sum_{k=-\infty}^{+\infty} \left| h[k] \right| = \infty \tag{2}$$

Mathematicians as well as theoretical physicists addressed the problem of determining how divergent a divergent infinite series is. In the research are of renormalization theory, researchers provided several results.The main purpose of this paper is to provide an alternate approach. The approach is summarized below

Let us consider a discrete LTI systems for which the impulse response sequence is bounded, i.e., there exists a real number M, such that

$$h[k] < M \tag{3}$$

for all values of $-\infty < k < \infty$. Now consider the related sequence $\hat{h}[\,]$, such that $\hat{h}[k] = \dfrac{h[k]}{M}$

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It is elementary to see that $\left|\hat{h}[k]\right| < 1$ for all k. Consider a causal, LTI system for which $\left|\hat{h}[k]\right| = 0$ for $k < 0$. Associate an infinite dimensional vector with the causal, unstable, discrete time LTI system $\left[\hat{h}[0], \hat{h}[1], \ldots\right]$.

Define N_p as the L_p -norm of the infinite dimensional vector

$$N_p = \left[\sum_{l=0}^{\infty}\left|\hat{h}[l]\right|^p\right]^{1/p} \quad for \;\; p \geq 1 \tag{4}$$

Lemma 1: *Np* is strictly monotone decreasing function for increasing values of ‘*p*’. Specifically for integral values of ‘*p*’ we have that

$N_1 > N_2 > \ldots$

Also, we have $\sum_{j=1}^{\infty} N_j < \infty$

Proof: For $p \geq 1$, $\left|\hat{h}[k]\right|^p < 1$ for all k. Thus, using basic properties of real numbers, we have that N_p is a strictly monotone decreasing function of p. Using D‘Alembert’s ratio test, it is inferred that $\sum_{j=1}^{\infty} N_j < \infty$

Lemma 2: Consider an infinite dimensional vector corresponding to a bounded, convergent scaled impulse response converging to a value less than one in magnitude. There exists an integer p such that the L_p -norm of such vector is finite (and hence for all larger values of).

Proof: $\left|\hat{h}[k]\right| < 1$ for all k. Hence, it is evident that $\left|\hat{h}[k]\right|^p$ approaches zero as p approaches ∞. Thus N_p converges to zero. Consequently, there exists an integer p_0 such that $N_p < \infty$ for all integer values of p larger than p_0.The above lemma effectively leads to an approach to quantify how unstable is an unstable LTI dynamical system. We have the following definition related to a new concept.Consider an unstable (BIBO) discrete time LTI system whose impulse response is bounded. Associate with it the scaled impulse response. Let the scaled impulse response converge to a value less than one in magnitude. Let it be denoted by $\hat{h}[.]$.

Definition: For such an unstable, bounded, convergent impulse response, there exists an *instability exponent* p_0 such that $N_q = \infty$ for all $q < p_0$ and $N_q < \infty$ for all $q \geq p_0$.Thus, based on above lemmas, we are able to associate a finite instability exponent with any unstable, discrete time LTI system.

A. Coarse-grain Ordering of Unstable Discrete-time LTI Systems

Using the L_p -norm of the impulse response sequence of discrete-time, LTI systems, we order them in a coarse manner.

Definition: Consider any two unstable LTI systems with impulse response sequences $\{C[.], D[.]\}$. Let the associated instability exponents be Γ_0, S_0. The system with impulse response $\{C[.]\}$ is highly unstable compared to the system with impulse response $\{D[.]\}$. if and only if $\Gamma_0 > S_0$.

It is possible to provide fine grain ordering of unstable systems. Details are avoided for brevity.

IV. Continuous Time, Unstable LTI Systems

We now consider the input-output description of a continuous time, LTI, causal system. It is well known that such a system is BIBO stable if and only if the impulse response is absolutely integrable i.e.,

$$\int_0^{\infty} |h(t)| dt < \infty$$

We now consider the class of unstable systems (in the BIBO sense) for which the impulse response is bounded (over the support) i.e.,

$$|h(t)| < k, \;\; t \in [0, \infty)$$

Now, define $\hat{h}(t) = \dfrac{h(t)}{K}$. Thus it is evident that

$$\left|\hat{h}(t)\right| < 1, \; t \in [0, \infty)$$

Now consider the L_p -norm of the function $\hat{h}(.)$, i.e.,

$$T_p = \left[\int_0^{\infty}\left|\hat{h}(t)\right|^p dt\right]^{1/p} \quad \text{for} \;\; p > 1$$

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The following lemma provides a basis for classifying unstable, continuous time, LTI systems.

Lemma 3: T_p is a strictly monotone decreasing function for increasing values of p. Specifically for integer values of p, we have that $T_1 > T_2 > T_3 > \ldots$

Proof: Follows from properties of the integral of a function of real variable.

Lemma 4: For every bounded impulse response function, there exists an integer r_0 such that $T_{r_0} < \infty$.

Proof: Avoided for brevity.

As in the case of discrete time LTI systems, the instability exponent enables ordering unstable, continuous time LTI systems.

V. State Space Formulation

In modern control theory, state-space formulation of discrete time as well as continuous time LTI systems is well studied. It is very well known that the input-output description of such systems can be easily obtained from the state-space description. Thus, all the results discussed in sections-3 and section-4 can be naturally be applied to such a state-space description.We are currently studying association of instability exponent directly with the state-space description.

VI. Future Research Work

In the journal version of this research paper, we investigate and propose results related to the following issues.

- Fine grain ordering of BIBO unstable discrete time as well as continuous time LTI systems.
- For first and second order LTI systems, we compute the time for the increasing (unbounded) impulse response to reach the lower and upper bounds.
- Using the modification of Routh-Hurwitz stability criteria, we propose an approach to investigate the relative instability of unstable LTI systems.
- In the case of robust stability, Kharitanov proposed investigating stability using four polynomials. We propose to investigate robust instability of LTI systems [4], using those polynomials.
- We formulate the following stability problem. The coefficients of characteristic polynomial are non-deterministic with associated probability distributions (PMF's or PDF's). Thus, we attempt to characterize the probabilistic robust instability of such systems. We also propose to investigate the concept of probabilistic robust stability of LTI systems.
- By a change of variable (real and imaginary part), the axes of S-plane can be translated to any point in the S-plane. The translated origin can be in the left half plane or right half plane. Using a modification Routh-Hurwitz criteria (as done in the case of relative stability), the number of zeros of characteristic polynomial in any quadrant (translated) is determined. Similar idea is executed for robust stability determination (using four polynomials specified byKharitanov criteria).
- Similarly, using a rotational transformation of 'S' variable, the zeros of the unstable system (characteristic polynomial) are localized.
- We propose to arrive at the notion of fine grain ordering of stable LTI systems.
- Researchers have investigated the stability of two/multi-dimensional LTI systems. Using those stability tests, relative/ robust stability could be investigated. In the spirit of research discussed in this paper, degree of instability of two/multi-dimensional systems will be investigated.

Conclusions

In this research paper, we first define some crude instability measures. Then, we associate discrete time as well as continuous time unstable LTI systems with instability exponents connected with their impulse response. We expect these instability measures to be of utility in applications.

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